Lecture 18

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1 Theoretical facts about image

Now we'll develop some theory about the image and its dimension and basis.

Let f be a linear function from V to U, and $\dim V = n$. Let's consider the kernel of f. We can find the basis of the kernel. Let it consist of r vectors, $e_1, e_2, \ldots, e_r$. Now we can use the procedure of extending to the basis to find the basis of the whole V which contains the vectors $e_1, e_2, \ldots, e_r$. Let we added vectors from e_{r+1} to e_n, so that the basis of V is

$$\overbrace{\underbrace{e_1,\ e_2,\ \ldots,\ e_r}_{\text{basis of } \operatorname{Ker} f},\ e_{r+1},\ \ldots,\ e_n}^{\text{basis of } V}$$

We know that the image of f is spanned by $f(e_1), f(e_2), \ldots, f(e_n)$. But $f(e_1),\ f(e_2),\ \ldots,\ f(e_r)$ are all equal to 0, because they belong to the kernel. So, the image is spanned by $f(e_{r+1})$, $f(e_{r+2}),\ \ldots,\ f(e_n)$. Now let's prove that they are linearly independent, and then it will be proved that they form a basis for the image.

To check their linear independence, let's write the linear combination which is equal to 0:

$$a_{r+1}f(e_{r+1}) + a_{r+2}f(e_{r+2}) + \cdots + a_n f(e_n) = \mathbf{0}.$$

This is the same as

$$f(a_{r+1}e_{r+1} + a_{r+2}e_{r+2} + \cdots + a_n e_n) = \mathbf{0}.$$

So, it follows that

$$a_{r+1}e_{r+1} + a_{r+2}e_{r+2} + \cdots + a_n e_n$$

belongs to the kernel of f. But the kernel is spanned by $e_1, e_2, \ldots, e_n$, so,

$$a_{r+1}e_{r+1} + a_{r+2}e_{r+2} + \cdots + a_n e_n = b_1 e_1 + \cdots + b_r e_r$$

or

$$a_{r+1}e_{r+1} + a_{r+2}e_{r+2} + \cdots + a_n e_n - b_1 e_1 - \cdots - b_r e_r = \mathbf{0}.$$

So, we got a zero linear combination of basis vectors which are independent. So, its coefficients are all equal to 0, and

$$a_{r+1} = a_{r+2} = \cdots = a_n = 0.$$

Thus, vectors are linearly independent. So, we see that the dimension of the image is equal to $n - r$, where $r = \dim \operatorname{Ker} f$. So, we got the following very important theorem:

Theorem 1.1. *If f is a linear function from V to U, and $\dim V = n$, then*

$$\dim \operatorname{Ker} f + \dim \operatorname{Im} f = n.$$

In our examples it was the case, since f was a function on 4-dimensional space, and $\dim \operatorname{Im} f + \dim \operatorname{Ker} f = 2 + 2 = 4$.

2 Application of image and kernel to matrices

In this section we'll try to use the theoretical results about dimensions of Image and Kernel to get a nice property of the rank of a matrix.

First, let's recall, that the following equality was proved:

$$\dim \operatorname{Im} f + \dim \operatorname{Ker} f = n, \tag{1}$$

where n is a dimension of a vector space in which the function is defined. Moreover, we derived the following formula for the dimension of the kernel:

$$\dim \operatorname{Ker} f = n - \operatorname{rk} A, \tag{2}$$

where A is a matrix of a linear function f. Comparing these two identities, we can easily prove, that

$$\dim \operatorname{Im} f = \operatorname{rk} A, \tag{3}$$

where A is a matrix of a linear function f. To compute the rank of a matrix, we had to transpose it to REF, and then the rank is equal to the number of nonzero rows. Now let's recall the algorithm of finding the dimension of the image. To find the dimension of the image, we had to write $f(e_1)$, $f(e_2)$, ..., $f(e_n)$ as ROWS of a matrix, than transpose it to REF, and the number of nonzero rows is the dimension of the image of f. But the matrix of a function is matrix with COLUMNS equal to $f(e_1)$, $f(e_2)$, ..., $f(e_n)$! So, the matrix of a function and the matrix from the algorithm of finding the dimension of the image are transposes of each other! So, because of the formula for the dimension of the image (3), we should get the same number of nonzero rows in REF for these two matrices. Thus we proved the following beautiful theorem:

Theorem 2.1. *For any matrix A*

$$\operatorname{rk} A = \operatorname{rk} A^\top.$$

We'll see how this proof works on an example.

Example 2.2. *Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear function such that*

$$f(x, y, z, u) = (x + y + 2z - u,\ 2x - 2y + z + u,\ 5x - 3y + 4z + u).$$

Now let's compute the values of this function on basis vectors:

- $e_1 = (1, 0, 0, 0)$, *then* $f(e_1) = f(1, 0, 0, 0) = (1, 2, 5)$.
- $e_2 = (0, 1, 0, 0)$, *then* $f(e_2) = f(0, 1, 0, 0) = (1, -2, -3)$.
- $e_3 = (0, 0, 1, 0)$, *then* $f(e_3) = f(0, 0, 1, 0) = (2, 1, 4)$.
- $e_4 = (0, 0, 0, 1)$, *then* $f(e_4) = f(0, 0, 0, 1) = (-1, 1, 1)$.

Thus, the matrix of f is

$$A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 2 & -2 & 1 & 1 \\ 5 & -3 & 4 & 1 \end{pmatrix}$$

Transforming it to REF

$$A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 2 & -2 & 1 & 1 \\ 5 & -3 & 4 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -4 & -3 & 3 \\ 0 & -8 & -6 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -4 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we see that the rank of A is equal to 2.

Now, to find the dimension of the image, we have to write the same vectors as rows and transpose matrix to REF. Then the dimension will be the number of nonzero rows:

$$A^\top = \begin{pmatrix} 1 & 2 & 5 \\ 1 & -2 & -3 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & -4 & -8 \\ 0 & -3 & -6 \\ 0 & 3 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 5 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We still got 2 nonzero rows! So, after transforming A and $A^\top$ to a REF, the numbers of their nonzero rows are equal.

3 Linear function as a vector space

Now we'll analyze some properties of linear functions.

Let's consider linear functions from the space V to the space U.

Zero The zero-function, i.e. such function $\mathbf{0}$ that for any vector v we get $\mathbf{0}$, i.e. $\mathbf{0}(v) = \mathbf{0}$ for all v, is linear:

$$\begin{aligned} f(v+u) = 0 = f(v) + f(u) \\ f(kv) = 0 = kf(v). \end{aligned}$$

Sum Let f and g be linear function, and let's consider their sum

$$(f+g)(v) \overset{\text{definition}}{=} f(v) + g(v)$$

Then it is a linear function:

$$\begin{aligned} (f+g)(v+u) &= f(v+u) + g(v+u) \\ &= f(v) + f(u) + g(v) + g(u) \\ &= (f+g)(v) + (f+g)(u). \end{aligned}$$

In the same manner we can check the multiplication property:

$$\begin{aligned} (f+g)(kv) &= f(kv) + g(kv) \\ &= kf(v) + fg(v) \\ &= k(f(v) + g(v)) \\ &= k(f+g)(v). \end{aligned}$$

So, we got that if functions f and g are linear, then their sum is a linear function as well.

Scalar multiplication Let f be a linear function, and k be a number. Let's consider the function

$$(kf)(v) \overset{\text{definition}}{=} kf(v).$$

Then this is still a linear function:

$$\begin{aligned} (kf)(v+u) &= kf(v+u) \\ &= k(f(v) + f(u)) \\ &= kf(v) + kf(u) \\ &= (kf)(v) + (kf)(u). \end{aligned}$$

In the same manner we can check the multiplication property:

$$\begin{aligned} (kf)(cv) &= kf(cv) \\ &= kcf(v) \\ &= ckf(v) \\ &= c(kf)(v). \end{aligned}$$

So, we got that if function f is linear, then function (kf) is linear for any number k.

So, since these 3 properties hold, we can deduce

Theorem 3.1. *Let V and U be vector spaces. The set of all linear functions $\{f : V \to U\}$ is a vector space.*

What is the dimension of it? Each linear function can be represented as a matrix, and if $\dim V = n$, and $\dim U = m$ then this matrix is $m \times n$-matrix. So, the dimension of the space of matrices is mn, and so is the dimension of the space of all linear functions from V to U.